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RESEARCH ARTICLE

Valuation before and after tax in the discrete time, finite state no arbitrage model

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Abstract We establish necessary and sufficient conditions for a linear taxation system to be neutral—within the multi-period discrete time "no arbitrage" model—in the sense that valuation is invariant to the exact sequence of tax rates, realization dates as well as immune to timing options attempting to twist the time profile of taxable income through wash sale transactions.

Keywords Tax neutrality · Mark-to-market valuation · Generalized linear taxation schemes · Wash sales

JEL Classification G12 · G13 · H20

> "In the study of investments, taxes are largely a source of embarrassment to financial economists."
>
> (Introduction to Dybvig and Ross 1986)

> "Accordingly, my approach in this chapter is to examine the restrictions on the income measurement rules applicable to financial instruments implied by the requirement that the rules be linear. ... Linearity is a desideratum of a tidy tax system."
>
> (Bradford 2000, p. 373–374)

1 Introduction

Tax considerations play a very important role in most real world investment decisions, whether financial or real. Tax issues, for example the use of tax shields, are also a cornerstone in corporate finance and capital structure theory.

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It is a remarkable fact, however, that the theoretical as well as the empirical asset pricing literature by and large ignores taxation issues and has little to say about the effects of different tax systems on asset demand functions and equilibrium market prices. Major textbooks on asset pricing theory and fixed-income analysis do not even have the word "taxation" or related topics in their index.

There may be a variety of reasons for why the asset pricing literature almost completely ignores taxation. Taxation issues are complicated and tend to destroy analytically attractive structures of asset pricing models. Taxation induces a certain individual element into return distributions and pricing relations, because return distributions and discount factors depend on the tax rules. If arbitrage opportunities after tax exist, some constraints like "limits to tax deductability", "limits to short positions" or other asset allocation restrictions are necessary in order to limit the extent to which such arbitrage opportunities can be exploited. The resulting market situation will be one marked by corner solutions and clientele effects which are inherently complicated to model.

However, if the tax system actually allows different agents with different tax rates—and maybe even different tax codes—to co-exist without experiencing arbitrage opportunities, one can justify the absence of tax considerations. Indeed, tax considerations are superfluous in the case of *neutral taxation systems*. A tax system is neutral if the implied tax rules lead to agreement about asset prices across investors subject to different tax treatment and also eliminate the profitability of portfolio dispositions solely made in order to avoid or defer tax payments.

The purpose of this paper is to examine the characteristics of such neutral taxation systems within the "no arbitrage" paradigm of financial economics. The analysis is carried out for linear taxation systems in order to stay within this paradigm and its characterization of linear pricing operators in terms of equivalent martingale measures.[1]

The paper provides a complete characterization of these systems and, hence, is able to identify when taxation issues can be ignored in asset pricing models. For simplicity we limit ourselves to the discrete time, finite state model. The aim for now is not to obtain as general a model as possible in terms of technical analysis, but rather to present the economic insight in a simple setting.

We ask and answer the following simple question: Given a scenario with different investors subject to linear and symmetric tax schedules, under what conditions is it possible that a set of asset prices can be a market equilibrium in the sense that all investors agree on these prices and that the resulting return distributions after tax do not give rise to arbitrage opportunities after tax? This property is our first criterion for a taxation system to be called neutral and we denote it as *valuation neutrality*.

The second criterion is that wash sale opportunities and tax timing options in a multi-period framework are eliminated. We denote this property as *holding period neutrality*, in line with the definition given in Auerbach (1991, p. 169):

[1] In a utility optimization model and even more in a complete equilibrium model arbitrage opportunities after tax can be bounded by having a progressive tax schedule. However, such models require the modeling of the individual's portfolio composition as a whole, see, e.g., Ross (1987), Dammon and Green (1987), Basak and Croituro (2001) and Basak and Gallmeyer (2003) for examples of such an approach. This is not the issue here. We restrict ourselves to an investigation of how far the no arbitrage paradigm and the consequent linearity of the pricing operator can be taken.

> "A realization-based tax system is holding-period neutral if it leads each investor in an asset to require a before-tax return having a certainty equivalent value that is not a function of the length of holding period or the asset's past pattern of return."

Clearly, the whole structure of prices and return distributions could change in a more fundamental sense as a result of introducing taxes in a model without taxes or as a result of changing tax rates or tax rules. Obvious channels for such consequences are wealth effects or wealth redistribution effects; additionally, taxation involves a risk sharing mechanism between the government and the tax-paying investor that may be affected when taxes are changed. However, such general equilibrium issues are not at stake here. Market prices are market prices for taxable investors as well as for tax free investors, and the point of departure is that market prices as well as certain types of investors with different tax rates exist. We want to examine—in a multi-period framework—whether such market prices can be consistent with heterogeneity across investors with respect to taxation in the sense that no arbitrage opportunities exist; in other words when a market equilibrium with "peaceful coexistence" among investors with different tax situations is possible. In particular, it is comforting to know when the usual before tax asset pricing relations can be valid in an economic setting where none of the investors are actually tax free.

Dealing with taxation in a multi-period framework raises the question about how to account for accrued capital income as taxable income. Most of the finance literature deals with a one-period framework, where investment takes place at the beginning of the period and capital income is earned and taxed at the end of the period. The seminal contributions by Schaefer, cf. Schaefer (1981, 1982a,b), belong to this class of models. Other examples are Dybvig and Ross (1986), Dammon (1987), Dammon and Green (1987), Ross (1987), Dermody and Prisman (1988) and Dermody and Rockafeller (1991). One-period models do not give rise to any problems with accruals. Capital income is taxed upon realization and accrued capital income does not exist in a one-period model. However, the conclusions from the one-period framework cannot be extrapolated to a multi-period framework without careful considerations.

A few papers deal with multi-period problems, but then capital income is taxed whenever it is realized—i.e., at the point in time where it has cash flow consequences. Seminal contributions in this direction encompass Constantinides (1983), Constantinides (1984) and Constantinides and Ingersoll (1984). Other examples are Dermody and Rockafeller (1995), Dammon and Spatt (1996), and Cadenillas and Pliska (1999). For such multi-period models it is well-known that the tax rules give rise to lock-in effects and that tax arbitrage problems are difficult to avoid unless some restrictions—short selling constraints, e.g.—are imposed.[2] Even so, taxable investors may be left with timing options. I.e., for any given series of market price movements a taxable investor can influence to his own advantage the timing of gains and losses as taxable income. A *wash sale* where assets are sold and immediately repurchased with the sole purpose of generating a tax deferral is a well-known example of such timing options.

[2] Cf. also footnote 1.

The basic results from the minimal possible no arbitrage model, the one-period binomial model, are stated in Sect. 2 as an introduction to the more general discussion in the rest of the paper. Following this we derive necessary and sufficient conditions for a linear and symmetric taxation system to be neutral in a multi-period framework. The main results are contained in Theorem 1 and the following Corollary 1. We provide an exhaustive characterization of the possible forms of linear, neutral taxation systems. The results show that a neutral taxation system calls for a split in the period by period income into a risk-free part and a risky part and a choice of two, almost separate, taxation mechanisms. One of these choices concerns the way risk-free income is taxed, and the result is identical to an imputed wealth tax. The other choice concerns the taxation of the risky part of the period by period income, i.e., the choice of a risk distribution mechanism between the government and the tax payer.

Interestingly enough, these conditions do not depend on any assumption about completeness of the market, although the necessity part rests on an innocent regularity assumption. The mark-to-market taxation system, the imputed wealth tax system as well as the pure cash flow taxation system due to Brown (1948) all fall into this category as specific examples. The taxation systems proposed by Auerbach (1991) and Auerbach and Bradford (2004) do so as well. Simultaneously, we are able to specify exactly the degree of uncertainty allowed for in the interest rate process as well as in the process describing the development of the tax rates over time. These issues receive little or no attention in the existing literature, where interest rates as well as tax rates are mostly assumed to be constant over time.

The paper is organized as follows. Section 2 introduces the linear tax system into the classical binomial model. Section 3 sets up the basic discrete time, discrete state space model. Section 4 contains the main results of the paper, necessary and sufficient conditions for tax neutrality, in Theorem 1 and Corollary 1. Finally, Sect. 5 shows an easy way to deal with partial realizations, dividend payments and capital injections as an alternative to tracing each transaction individually. Summary and conclusions are found in Sect. 6.

Proofs are found in Appendix A. Appendix B completes some details of the derivations in Sect. 5.

2 The one-period binomial model

Consider the usual setup for the binomial model with one risky asset and a risk-free asset with interest rate r. In this setup the fundamental pricing relation is stated in terms of the equivalent martingale measure $(q, 1-q)$ as

$$(1+r)S_0 = qS_u + (1-q)S_d. \tag{1}$$

The exact value of q, the current asset price S_0 as well as the price process (S_u, S_d) are results of a general equilibrium. They will all depend on investor characteristics such as time preference, the degree of risk aversion and the distribution of wealth across investors—more generally, the marginal utility of the individual investor's intertemporal utility function.

Consider a given investor in this market equilibrium and denote the tax rate for this investor by T; this tax rate is assumed to be the same for all assets under consideration and is applied linearly and symmetrically to a tax base comprised of net earnings. I.e., losses are fully and unconditionally deductible with full tax consequences under all circumstances. Then the following manipulation of Eq. (1) into Eq. (3) reflects the taxation on a net earnings basis:[3]

$$(1-T)(1+r)S_0 = (1-T)\left[qS_u + (1-q)S_d\right] \quad \Leftrightarrow \tag{2}$$

$$\begin{aligned}(1+r(1-T))S_0 &= q(1-T)S_u + (1-q)(1-T)S_d + TS_0\\ &= q\left[S_u - T(S_u - S_0)\right] + (1-q)\left[S_d - T(S_d - S_0)\right].\end{aligned} \tag{3}$$

The terms in [ ]-brackets in (3) are the payments after tax to the investor under such a linear tax rule; and we will denote such after tax magnitudes in general by the superscript $()^{a.t.}$. Hence

$$(1+r(1-T))S_0 = qS_u^{a.t.} + (1-q)S_d^{a.t.} \quad \Leftrightarrow \tag{4}$$

$$\begin{aligned}S_0 &= [1+r(1-T)]^{-1}\left[qS_u^{a.t.} + (1-q)S_d^{a.t.}\right]\\ &\equiv \left[1+r^{a.t.}\right]^{-1}\left[qS_u^{a.t.} + (1-q)S_d^{a.t.}\right].\end{aligned} \tag{5}$$

The pricing relation (5) is also valid for the risk-free asset with $S_0 = 1, S_u = S_d = 1+r$ and $S_u^{a.t.} = S_d^{a.t.} = 1 + r^{a.t.}$. Thus, (5) is valid for this investor on an after tax basis if and only if (1) is valid on a before tax basis.[4]

Any investor in a no arbitrage equilibrium prices assets in terms of an equivalent martingale measure and a discount factor, which in combination forms the investor's Arrow-Debreu prices. In an after tax setting these Arrow-Debreu prices must apply to the after tax value of any asset in order to give the market value S_0. In a standard binomial setting as above, cf. (1), the equivalent martingale measure $(q, 1-q)$ is uniquely determined. This is also the case in the after tax setting in (5). Observe that these two measures are identical. The taxation only affects the discount factor, leaving the relative Arrow-Debreu price across the two states unaffected.

However, there is no need for a before tax pricing relation. If one investor in the market equilibrium with tax rate T prices assets as shown in (5), then, cf. Lemma 1 below, any other investor in the market equilibrium with a different tax rate $\widehat{T}$ also prices assets in accordance with (5).

Example 1 Let $S_0 = 100$, $S_u = 110$ and $S_d = 95$. With an interest rate of 5% the martingale probability is $q = 2/3$.

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[3] In this simple one-period setup there is no difference between taxation in accordance with the mark-to-market valuation principle and taxation in accordance with the realization principle.

[4] This result is also found in Cox and Rubinstein (1985, p. 271–274), as a special case. However, no multi-period analysis after tax is found in Cox and Rubinstein (1985).

The corresponding after tax payments for an investor with a tax rate $T = 40\%$ are $S_u^{a.t.} = 106$, $S_d^{a.t.} = 97$ and $r^{a.t.} = 3\%$. Then

$$100 = (1.03)^{-1}\left[\frac{2}{3}106 + \frac{1}{3}97\right]. \tag{6}$$

In general

$$\begin{aligned}100 &= (1+0.05(1-T))^{-1}\left[\frac{2}{3}(110-10T) + \frac{1}{3}(95+5T)\right]\\ &= (1.05-0.05T)^{-1}\left[105-5T\right].\end{aligned} \tag{7}$$

The taxation of capital gains and losses as shown in (5) induces one particular type of risk sharing among the investor and the tax authorities which is compatible with valuation neutrality and the absence of arbitrage on an after tax basis. However, other linear taxation schemes and induced risk sharing schemes with this property are possible. One extreme case would be that the investor is allowed to earn exactly the risk-free rate of return after tax in both states. In the above Example 1 the after tax result would be 103 in both states and all uncertainty about the return on the risky investment will be borne by the tax authorities. Hence, the name "government takes all risk" is appropriate for this taxation system which is obviously neutral.

Another example is to tax the investor the amount rTS_0, irrespective of the actual outcome of the risky investment. This would leave the investor with the same risk after tax as before tax in the sense that the spread between the two outcomes as well as the spreads relative to the possible risk-free rate of return are the same on an after tax basis as on a before tax basis. This is known as an imputed wealth tax and is also a neutral taxation principle—it could also be termed the "government takes no risk" principle.

A third example is to tax the investor by applying the tax rate $1-(1+r^{a.t.})/(1+r)$ to the final value S_1 of the investment. In this way the gross return after tax to the investor would the fraction $(1+r^{a.t.})/(1+r)$ of the before tax value of the investment, irrespective of whether the investor has gained or lost on his investment. This fraction is the same fraction of the before tax value as the investor retains by investing in a bank account, and it leaves more risk with the investor than the mark-to-market valuation principle, but less than the imputed wealth tax system. This is a neutral taxation system because, irrespective of the size of T, we have

$$(1+r)S_0 = qS_u + (1-q)S_d \;\Leftrightarrow\; \left(1+r^{a.t.}\right)S_0 = q\frac{1+r^{a.t.}}{1+r}S_u + (1-q)\frac{1+r^{a.t.}}{1+r}S_d. \tag{8}$$

All of the above taxation schemes can be obtained by applying the following more general procedure:

- rewrite (1) by moving $(1+r)S_0$ to the right hand side
- multiply it by any positive number K

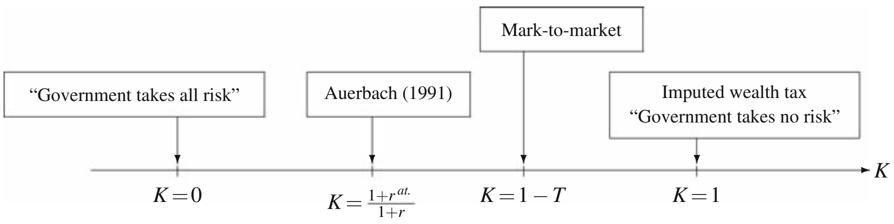

Fig. 1 Varying degrees of risk taking in the tax system

– add $(1 + r^{a.t.})S_0$ on both sides

in order to obtain the after tax equivalent to (1):

$$(1+r)S_0 = qS_u + (1-q)S_d \quad \Leftrightarrow$$
$$0 = q\,[S_u - (1+r)S_0] + (1-q)\,[S_d - (1+r)S_0] \quad \Leftrightarrow \tag{9}$$
$$(1+r^{a.t.})S_0 = q\left[K(S_u - (1+r)S_0) + (1+r^{a.t.})S_0\right]$$
$$+(1-q)\left[K(S_d - (1+r)S_0) + (1+r^{a.t.})S_0\right]. \tag{10}$$

The parameter K is a measure of the degree of risk sharing with the government. Equation (10) is trivially true for the risk-free asset, independent of K, and it is also trivially true for $K = 0$, which corresponds to the taxation principle "government takes all risk", where the investor receives $(1 + r^{a.t.})S_0$ in both states. For $K = 1 - T$ it is the mark-to-market valuation taxation, as shown in (3), where the investor receives $(1 - T)S_1 + TS_0$. The case $K = \frac{1+r^{a.t.}}{1+r}$ gives the relation (8); it is identical to the "retrospective capital gains taxation" system from Auerbach (1991) which is considered in more detail in Sect. 4. When $K = 1$ the imputed wealth taxation system with tax payment $(r - r^{a.t.})S_0 = rTS_0$ in both states and no risk sharing with the government arises (Fig. 1).

A common feature across the choice of K is that the investor earns an expected return after tax under the equivalent martingale measure equal to the after tax rate of interest $r^{a.t.}$. Another common feature is that the tax to be paid is a linear function of the values (S_0, S_1), and when applied to the risk-free asset the return in both states is $r^{a.t.} \equiv r(1 - T)$. However, it is only the taxation on a net income basis—as shown in Eqs. (4)–(5)—that has the property that the tax to be paid is a linear function of the net earning $S_1 - S_0$; the other linear schemes require weights to be applied to S_1 and S_0 separately.

The findings for this simple one-period scenario and a given market equilibrium are summarized in Lemma 1.

Lemma 1 *The pricing relation* (10) *after tax for some investor*

– *subject to a linear and symmetric taxation*
– *with taxation given by* $K \neq 0$ *and* $r^{a.t.} > -1$

is equivalent to the pricing relation (10) *after tax for some other investor*

– *subject to a linear and symmetric taxation*

– *with taxation given by* $\hat{K} \neq 0$ *and* $\hat{r}^{a.t.} > -1$

Furthermore, the set of attainable claims after tax is the same for all such investors. With the exception of $K = 0$ *this set is identical to the set of attainable claims before tax.*

Proof The first part of the lemma is obvious from (9)–(10). For the second part, see Appendix A. □

3 The multi-period discrete time model

One-period models are void of problems with accrued capital gains, wash sale opportunities and timing options. The only concern for neutrality of a taxation system in a one-period model is valuation neutrality.

However, the basic insight from the one-period model is useful in order to study neutrality in a multi-period framework. In a multi-period framework it is necessary to define taxation rules for accrued capital gains. In addition to valuation neutrality it is also necessary to examine whether the tax rules are holding period neutral.

In the following we employ the standard setup for the discrete time model with a finite set of possible realizations.[5] I.e., we are given a probability space $(\Omega, \mathscr{F}, P)$ where Ω is a finite set of states, $\Omega \equiv \{1, 2, \ldots, M\}$, and a filtration $\{\mathscr{F}_t\}_{t=0}^{t=n}$ to be specified shortly. Without loss of generality we take $\mathscr{F}$ to be the discrete algebra:

$$\mathscr{F}_n \subseteq \mathscr{F} = 2^{\Omega}, \qquad \mathscr{F}_0 = \{\emptyset, \Omega\}, \qquad P_\omega > 0 \quad \forall \omega \in \Omega.$$

In terms of this notation, $\{0, 1, 2, \ldots, n\}$ denotes the set of time indices, where trading in financial assets is allowed and where it is possible to either withdraw money from the portfolio in order to consume or to invest additional equity into the portfolio.

The financial market has K traded assets. These assets are assumed not to pay dividends. Dividends are easily incorporated, but they merely contribute to a more complicated notation in the presentation of the basic arguments.[6] The prices at time t of these assets are denoted by the vector $S_t \in \mathbb{R}^K$. We take the filtration $\{\mathscr{F}_t\}_{t=0}^{t=n}$ as the "natural" filtration, i.e., the filtration generated by the K-dimensional price process S_t.

One of these assets is the bank account. This is denoted by B_t and develops as

$$B_{t+1} = (1 + r_t)B_t, \quad B_0 \equiv 1 \tag{11}$$

where r_t is the one-period interest rate valid at time t for the period from time t until time $t + 1$. The process r_t may well be stochastic, but it is required to be adapted to $\mathscr{F}_t$. Hence, the bank account is a predictable process, i.e., B_t is $\mathscr{F}_{t-1}$-measurable.

[5] The basics of the discrete time model is spelled out in numerous places. See, e.g., Pliska (1997, Chap. 3), for a good introduction or Jensen and Nielsen (1996) for a comprehensive development of the discrete time model with a finite sample space. None of the conclusions in the following depend in any essential manner on the assumption that the sample space is finite, cf. e.g., Kabanov and Kramkov (1995). However, some technicalities are simpler with this assumption with no loss of economic insight.

[6] Dividends are discussed shortly in Sect. 5.

A trading strategy $(\theta^1, \theta^2, \ldots, \theta^n)$ is also a predictable process with values in $\mathbb{R}^K$. θ^j is the portfolio position held from time $j-1$ until time j. The exiting value at time $j-1$ is given by the inner product $\theta^j S_{j-1}$. The entering value at time j is similarly given by the inner product $\theta^j S_j$.

Unless otherwise stated we assume in the following that the trading strategies considered are self-financing on a before tax basis. For such trading strategies the notational convention $\theta^0 \equiv \theta^1$ is natural; the amount of wealth for the first exiting portfolio $\theta^1 S_0$ chosen at time 0 is the amount of money that exists when entering into the first period under consideration. In Sect. 5 we treat the case where the portfolio is not self-financing, i.e., where the investor can make both withdrawals from the portfolio and capital injections into the portfolio at any point in time.

For any predictable trading strategy relation (12) is merely a book keeping identity:

$$\frac{\theta^n S_n}{B_n} = \theta^0 S_0 + \sum_{t=1}^{n} \theta^t \left(\frac{S_t}{B_t} - \frac{S_{t-1}}{B_{t-1}} \right) + \sum_{t=1}^{n} \left(\theta^t - \theta^{t-1} \right) \frac{S_{t-1}}{B_{t-1}}. \tag{12}$$

By splitting the two sums in (12) at a particular time j this relation can be rewritten as

$$\begin{aligned}
&\frac{\theta^n S_n}{B_n} + \sum_{t=j}^{n-1} \left(\frac{\left(\theta^t - \theta^{t+1}\right) S_t}{B_t} \right) \\
&= \theta^0 S_0 + \sum_{t=1}^{j} \theta^t \left(\frac{S_t}{B_t} - \frac{S_{t-1}}{B_{t-1}} \right) + \sum_{t=1}^{j} \left(\theta^t - \theta^{t-1} \right) \frac{S_{t-1}}{B_{t-1}} + \sum_{t=j+1}^{n} \theta^t \left(\frac{S_t}{B_t} - \frac{S_{t-1}}{B_{t-1}} \right) \\
&= \frac{\theta^j S_j}{B_j} + \sum_{t=j+1}^{n} \theta^t \left(\frac{S_t}{B_t} - \frac{S_{t-1}}{B_{t-1}} \right).
\end{aligned} \tag{13}$$

As a consequence of the "no arbitrage" pricing relation under any equivalent martingale measure we have that:

$$\frac{S_j}{B_j} = E^Q \left[\frac{S_{j+1}}{B_{j+1}} \,\middle|\, \mathscr{F}_j \right]. \tag{14}$$

And by the law of iterated conditional expectations and the fact that $\theta^j \in \mathscr{F}_{j-1}$ we arrive at the following result from taking conditional expectations in (13):

$$\frac{\theta^j S_j}{B_j} = E^Q \left[\frac{\theta^n S_n}{B_n} + \sum_{t=j}^{n-1} \frac{(\theta^t - \theta^{t+1}) S_t}{B_t} \,\middle|\, \mathscr{F}_j \right]. \tag{15}$$

Equation (15) expresses the valuation property that the value of a trading strategy at any point in time, expressed in terms of the bank account as the numeraire, is the conditional expected present value under Q of the realization value at the horizon time n, corrected for the net costs in present value terms of changing the portfolio at intermediate times $1, 2, \ldots, n-1$. For a self-financing trading strategy, $\theta^t S_t = \theta^{t+1} S_t$, so the summation term in (15) disappears, rendering it identical to (14).

4 Generalized linear taxation

In a multi-period setting the typical objections to a tax system based on earnings measurement unrelated to actual realizations, like, e.g., mark-to-market valuation based taxation, are concerned with the liquidity needs of the investor and the informational requirements caused by repeated assessments of illiquid capital assets. For this reason there has been some interest in the literature to solve the liquidity problem by constructing *realization based* taxation systems that eliminate wash sale opportunities and induce holding period neutrality. Furthermore, a realization based taxation system that only needs to observe *net cash flows* in and out of the portfolio to form the tax base will solve the information problem.

The class of generalized linear taxation schemes under consideration in this paper are characterized by a tax account which measures the investor's net liability towards the tax system at any point in time. The tax account is a function of the investment strategy θ, and its value at time t is denoted by A_t^θ. For unrealized positions the investor can carry this net liability forward in time, but in an interest bearing manner analogous to the balance on a bank account. The idea of such a tax account as a means to avoid liquidity problems dates back at least to Vickrey (1939).[7] We aim at characterizing in an exhaustive manner the set of linear taxation rules that eliminate tax arbitrage opportunities and guarantee valuation neutrality as well as holding period neutrality.

We start by specifying the value of this tax account as a linear function of the sequence of observed asset values $\theta^j S_j$. When the tax liability is a linear function of observed asset values $\theta^j S_j$, so is the investor's portfolio value after correction for the tax liability. The coefficients in these two linear functions are denoted by $K_{t,j}^0$ as follows:

$$A_t^\theta = \left(1 - K_{t,t}^0\right) \theta^t S_t - \sum_{j=0}^{t-1} K_{t,j}^0 \theta^j S_j \quad \text{(tax account)} \tag{16}$$

$$\theta^t S_t - A_t^\theta = \sum_{j=0}^{t} K_{t,j}^0 \theta^j S_j \quad \text{(investor's wealth after tax).} \tag{17}$$

We will refer to this way of representing the taxation system as *stock value based*. The superscript in $K_{n,j}^0$ refers to the starting date 0 as the last date where a transaction took place. We define $K_{0,0}^0 \equiv 1$ as the natural initial condition, i.e., the tax account has initial

[7] However, the primary purpose of Vickrey's suggestions were more inspired by a desire to even at the tax base and the tax payments over time than a desire to prove neutrality properties in a stringent manner. Progressive taxation schedules, loss carry forward provisions and similar aspects of the tax code are themselves arguments for even out procedures, and Vickrey cites earlier real life examples of implementations of such procedures.

value zero.[8] In technical terms we require these coefficients to be predictable processes, i.e., they are known at least one period ahead at any time where a portfolio revision is allowed to take place. I.e., $K^0_{t,j} \in \mathscr{F}_{t-1}$ for $j = 0, 1, \ldots, t$. Since $B^{a.t.}_t \in \mathscr{F}_{t-1}$ we also have that $K^0_{t,j}/B^{a.t.}_t \in \mathscr{F}_{t-1}$ for $j = 0, 1, \ldots, t$.

There is an equivalent representation that focuses on the net earnings period by period. The equivalence is a simple consequence of the rules for partial summation,[9] and we denote this representation as *earnings based*:

$$\sum_{j=0}^{n} K^0_{n,j}\theta^j S_j = k^0_{n,0}\theta^0 S_0 + \sum_{j=1}^{n} k^0_{n,j}\left(\theta^j S_j - \theta^{j-1} S_{j-1}\right) \tag{18}$$

where

$$k^0_{n,n} = K^0_{n,n}, \qquad k^0_{n,j} = k^0_{n,j+1} + K^0_{n,j}, \quad j = 0, 1, \ldots, n-1. \tag{19}$$

For some linear tax systems, e.g., taxation in accordance with the mark-to-market valuation principle, it is natural to think in terms of the earnings based representation. For other linear taxation systems, e.g., the imputed wealth tax, it is natural to think in terms of the stock value based representation. However, as the above equivalent representations show, one can choose either way in all cases.

Valuation neutrality requires that the after tax realization value at any date t, $t = 0, 1, 2, \ldots, n-1$, expressed in terms of the numeraire $B^{a.t.}_t$, is equal to the time t value of the portfolio after tax at any future date up to and including the horizon n. I.e., for an appropriate martingale measure Q the following must hold for any pair of dates (t, m), $m \leq n$, $t = 0, 1, 2, \ldots, m-1$:

$$\frac{\sum_{j=0}^{t} K^0_{t,j}\theta^j S_j}{B^{a.t.}_t} = E^Q\left[\frac{\sum_{j=0}^{m} K^0_{m,j}\theta^j S_j}{B^{a.t.}_m} \,\middle|\, \mathscr{F}_t\right] \quad \Leftrightarrow \tag{20}$$

$$0 = E^Q\left[\sum_{j=0}^{t}\left(\frac{K^0_{m,j}}{B^{a.t.}_m} - \frac{K^0_{t,j}}{B^{a.t.}_t}\right)\theta^j S_j + \sum_{j=t+1}^{m}\left(\frac{K^0_{m,j}}{B^{a.t}_m}\right)\theta^j S_j \,\middle|\, \mathscr{F}_t\right]. \tag{21}$$

In Theorem 1 we state that for any no arbitrage pricing relation before tax and associated equivalent martingale measure Q, the conditions in (22) are sufficient conditions for (21) to be fulfilled for the same martingale measure:

$$\frac{K^0_{m,j}}{B^{a.t}_m} = \frac{K^0_{t,j}}{B^{a.t}_t} \quad \text{for } j = 0, 1, \ldots, t-1, \;\; K^0_{t,t} = B^{a.t.}_t \sum_{j=t}^{m} \frac{K^0_{m,j}}{B^{a.t.}_m}\frac{B_j}{B_t}. \tag{22}$$

[8] This means that the relations with the tax authorities are cleared at time 0 and that no relevant prehistory exists. This is generalized in Sect. 5.

[9] The discrete analogue to integration by parts.

These conditions are also necessary provided that the stochastic dynamics of asset prices is sufficiently rich in the sense that non-trivial trading strategies exist; i.e., trading strategies such that $\theta^u S_u/B_u$ is never constant for any two adjacent date-events.[10] As a matter of fact this a very modest requirement.

The first part of the conditions in (22) implies that the tax consequences that would have occurred at time m, and ultimately at the horizon n, due to time periods already lapsed at time t are invariant to the choice of realization date; hence also invariant to timing options and attempted wash sales. The difference is solely due to the discounting effect of paying taxes later rather than sooner, whereas the risk sharing mechanism between the investor and the government is the same. Or, stated differently, the time 0 present value of the after tax proceeds, when applying the value of the bank account after tax as the numeraire, is independent of the choice of realization date. One consequence of this is that the first sum in (21) is eliminated, except for the term $j = t$. Another consequence is that $K^0_{m,j}/B^{a.t}_m \in \mathscr{F}_{t-1}$ for any $t > j$. Hence, $K^0_{m,j}/B^{a.t}_m \in \mathscr{F}_j$.

The second part of the conditions in (22) can be explained by the following financial engineering experiment. Assume that the investor decides to *realize* the position at time t, clear the tax account and afterwards invest the proceeds in a conventional bank account until time m. The result of this strategy is

$$\left(\sum_{j=0}^{t} K^0_{t,j}\theta^j S_j\right)\frac{B^{a.t.}_m}{B^{a.t.}_t}. \tag{23}$$

If the investor alternatively claims that the portfolio is *rebalanced* and the reinvestment is done in a *synthetic bank account*, an asset developing in value according to $\theta^t S_t\,(B_u/B_t)$ and subject to taxation in accordance with the general linear scheme applied to the sequence of values $\theta^t S_t\,\left(B_j/B_t\right),\ j = t{+}1,\ldots,m$, the final result will be

$$\sum_{j=0}^{t} K^0_{m,j}\theta^j S_j + \sum_{j=t+1}^{m} K^0_{m,j}\theta^t S_t\frac{B_j}{B_t}. \tag{24}$$

The terms corresponding to $j = 0, 1, 2, \ldots, t-1$ are identical, given the first set of conditions in (22). The latter part of (22) equates the final result of these two economically fully identical investment strategies.

It is worth mentioning that these restrictions are independent of whether the market is complete or not, whether the short term interest rate is stochastic or not and whether the taxation parameters $K^0_{t,j}$ are stochastic or not. The only requirement on the interest rate process and the taxation parameters is that they are predictable.

The second part of the conditions in (22) places strong restrictions on the possible stochastic variation in the taxation parameters and their interplay with the possible stochastic variation in interest rates and tax rates: The sum of $\mathscr{F}_j$-measurable variables, $j = t, t+1, \ldots, m-1$, must be an $\mathscr{F}_{t-1}$-measurable variable. The mark-to-market

[10] Alternatively stated, the splitting index is always at least two.

principle is a good example of this. Despite the possible stochastic nature of $K^0_{n,j}$ the sum of the r.h.s. in (22) is $(1 - T_{t-1})$ as shown in Table 1 below.

The conditions in (22) are mostly relative restrictions on the model parameters. However, for $t = 0$ the restriction $K^0_{0,0} \equiv 1$ is sufficient to determine the absolute values of the parameters; i.e.,

$$1 = \sum_{j=0}^{n} \frac{K^0_{n,j} B_j}{B^{a.t.}_n} \quad \Leftrightarrow \quad B^{a.t.}_n = \sum_{j=0}^{n} K^0_{n,j} B_j. \tag{25}$$

Again, the restriction in (25) is an invariance property between investing in a conventional bank account over the time horizon $j = 0, 1, \ldots, n$ versus investing in a synthetic bank account subject to taxation in accordance with the linear taxation parameters $K^0_{n,j}$ applied to the sequence of asset values $B_0, B_1, B_2, \ldots, B_n$.

Next consider a wash sale operation at time t. We assume that the self-financing portfolio policy is re-established by taking a short position in the bank account to pay the tax liability A^θ_t and restart the linear tax system at time t with new coefficients $K^t_{n,j}$. Then—analogous to (21)—holding period neutrality is guaranteed whenever

$$\frac{\theta^t S_t - A^\theta_t}{B^{a.t.}_t} = E^Q\left[\frac{\sum_{j=t}^{s} K^t_{s,j}\theta^j S_j - A^\theta_t\left(B^{a.t.}_s / B^{a.t.}_t\right)}{B^{a.t.}_s} \,\middle|\, \mathscr{F}_t\right], \quad s = t+1, \ldots, n \Leftrightarrow \tag{26}$$

$$\theta^t S_t = E^Q\left[\frac{\sum_{j=t}^{s} K^t_{s,j}\theta^j S_j}{\left(B^{a.t.}_s / B^{a.t.}_t\right)} \,\middle|\, \mathscr{F}_t\right], \quad s = t+1, \ldots, n. \tag{27}$$

Assume that we reset the clock by adjusting all period indices by t. The bank account expression $\left(B^{a.t.}_s / B^{a.t.}_t\right)$ is then just the accumulation after tax in a bank account over the time period $[0, s-t]$. Then (27) is equivalent to the requirement

$$\theta^0 S_0 = E^Q\left[\frac{\sum_{j=0}^{s} K^0_{s,j}\theta^j S_j}{B^{a.t.}_s}\right], \quad s = 1, \ldots, n-t \tag{28}$$

which is nothing but the fundamental requirement for a neutral taxation system. This shows that if (i) the clock as well as the bank account is reset after a wash sale and (ii) the tax system is restarted we will not be able to make any tax arbitrage gains. Since the tax account is reset to value zero it is also possible to "start from scratch" and change to any another neutral system upon realization.[11]

The main results for such generalized linear taxation systems are now stated in Theorem 1, and this is followed by examples of specific models.

Theorem 1 *Linear and symmetric neutral taxation schedules in the discrete time, finite state "no arbitrage" pricing model are exhaustively characterized as follows:*

[11] Section 5 outlines one way to implement this recalculation of coefficients that is particularly well suited for partial realizations.

1. *For any given "no arbitrage" equilibrium before tax and related equivalent measure Q the parameter restrictions for any pair of dates (t, m), $t = 0, 1, 2, \ldots, m-1$, $m \leq n$:*

$$\frac{K^0_{m,j}}{B^{a.t}_m} = \frac{K^0_{t,j}}{B^{a.t}_t} \quad \text{for } j = 0, 1, \ldots, t-1, \qquad \frac{K^0_{t,t}}{B^{a.t.}_t} = \sum_{j=t}^{m} \frac{K^0_{m,j}}{B^{a.t.}_m} \frac{B_j}{B_t}, \, K^0_{0,0} = 1 \tag{29}$$

are sufficient to produce the analogous "no arbitrage" equilibrium after tax:

$$\frac{\sum_{j=0}^{t} K^0_{t,j} \theta^j S_j}{B^{a.t.}_t} = E^Q \left[\frac{\sum_{j=0}^{m} K^0_{m,j} \theta^j S_j}{B^{a.t.}_m} \,\middle|\, \mathscr{F}_t \right]. \tag{30}$$

2. *Assume that the stochastic dynamics of asset prices is sufficiently rich in the sense that non-trivial self-financing trading strategies exist, i.e., self-financing trading strategies such that $\theta^u S_u / B_u$ is never constant across adjacent date-events. Then valuation neutrality for any given taxable investor in the sense that (30) is fulfilled for some equivalent martingale measure Q and any pair of dates (t, m), $t = 0, 1, 2, \ldots, m-1$, $m \leq n$, requires the parameter conditions in (29) as necessary conditions. If, furthermore, $K_{tt} \neq 0 \; \forall t$ then valuation neutrality for some taxable investor with some equivalent martingale measure Q implies valuation neutrality for any other investor with the same equivalent martingale measure, but other taxation parameters satisfying (29).*
3. *Define the $\mathscr{F}_j$-measurable variables ψ_j and the $\mathscr{F}_{t-1}$-measurable variables ϕ_t as follows:*

$$\begin{aligned} \psi_j &\equiv \frac{K^0_{n,j}}{B^{a.t.}_n}, \quad j = 0, 1, \ldots, n-1, \\ \phi_t &\equiv \frac{K^0_{t,t}}{B^{a.t.}_t}, \quad t = 1, 2, \ldots, n, \qquad \phi_0 \equiv 1, \qquad \phi_{n+1} \equiv 0. \end{aligned} \tag{31}$$

Then the following relation determines the "off-diagonal" taxation parameters $K_{n,j}$ in terms of the "diagonal" ones:

$$\psi_j = \phi_j - (1 + r_j)\phi_{j+1} \tag{32}$$

4. *The above conditions are sufficient to ensure holding period neutrality, i.e., the elimination of any wash sale opportunity at time t. One can restart the tax system after resetting the bank accounts in accordance with the parameters $K^0_{t,j}$, but it is also possible to switch to any other neutral tax system after a realization or an attempted wash sale.*
5. *Any convex combination of two linear and symmetric neutral taxation schedules is again a linear and symmetric taxation schedule.*

6. *The set of attainable claims after tax is the same for all investors for which* $\phi_t \neq 0$ $\forall t$*, and this set is identical to the set of attainable claims before tax for all investors for which* $\phi_t \neq 0$ $\forall t$.

Proof Part of the proof has already been given above. The completion of the proof is found in Appendix A. □

By straightforward manipulations, using (20) and (29), we have the following corollary to Theorem 1:

Corollary 1 *A neutral taxation system is given by its diagonal elements* K^0_{tt} *and the wealth dynamics for an unrealized portfolio position is given by*

$$\begin{aligned}&\left(\theta^{t+1}S_{t+1} - A^\theta_{t+1}\right) - (1 + r_t(1 - T_t))\left(\theta^t S_t - A^\theta_t\right)\\&\quad = K^0_{t+1,t+1}\left[\theta^{t+1}S_{t+1} - (1 + r_t)\theta^t S_t\right].\end{aligned} \tag{33}$$

The tax account develops according to

$$A^\theta_{t+1} = (1 + r_t(1 - T_t))A^\theta_t + r_t T_t \theta^t S_t + \left(1 - K^0_{t+1,t+1}\right)\left(\theta^{t+1}S_{t+1} - (1 + r_t)\theta^t S_t\right). \tag{34}$$

According to Corollary 1 any generalized linear taxation system decomposes into two elements:

1. an imputed wealth tax at tax rate T_t, which can be interpreted as a tax on the change in wealth due to "passage of time", $r_t\theta^t S_t$.
2. a tax on the risky part of the pre-tax earnings, i.e., on pre-tax earnings net of the wealth change due to "passage of time", with tax rate $1-K^0_{tt}$.

Hence, the diagonal elements K^0_{tt}, together with the way risk-free income is taxed, is sufficient to describe the tax system. The magnitudes $1-K^0_{tt}$ serve the role of tax rates applicable to the risky income component, whereas $r_t T_t \theta^t S_t$ is the imputed wealth tax. In terms of economic interpretation, the tax authorities or the legislators can choose a *risk distribution* mechanism between the government and the investor by choosing the diagonal tax parameters $K^0_{t,t}$. Having done this the "off-diagonal" tax coefficients follow from the derived conditions, and the rest of the tax structure is determined by the development of the bank account after tax, i.e., determined by the way risk-free returns are taxed over time.

Remark 1 Note that a neutral taxation system does not allow for differential taxation of, say, bonds and stocks as such. But it does allow for differential taxation of a risk-free part of the return from any asset versus the remaining risky part of the return.

Table 1 identifies the parameters $K^0_{n,j}$ for the particular models referred to in this paper for the stock value based representation of the linear taxation mechanism. Table 2 identifies the corresponding parameters $k^0_{n,j}$ for the earnings based representation of the linear taxation mechanism.

Table 1 Parameterizations for six special models, stock value based

	$K^0_{n,j},\ j=0,1,2,\ldots,n-1$	$K^0_{n,n}$
Mark-to-market	$\left[T_j - T_{j-1}\left(1+r^{a.t.}_j\right)\right]\frac{B^{a.t.}_n}{B^{a.t.}_{j+1}}\ T_{-1}\equiv 0$	$1-T_{n-1}$
Auerbach (1991)	0	$\frac{B^{a.t}_n}{B_n}$
Auerbach and Bradford (2004)	$1_{\{j=0\}}HB^{a.t}_n,\ H\in[0,1]$	$(1-H)\frac{B^{a.t}_n}{B_n}$
Brown (1948)	$1_{\{j=0\}}TB_n,\ B^{a.t.}_n=B_n$	$1-T$
Imputed wealth tax	$-r_jT_j\frac{B^{a.t.}_n}{B^{a.t.}_{j+1}}$	1
Government takes all risk	$1_{\{j=0\}}B^{a.t}_n$	0

Table 2 Parameterizations for six special models, earnings based

	$k^0_{n,j},\ j=1,2,\ldots,n$	$k^0_{n,0}$
Mark-to-market	$1-T_{j-1}\frac{B^{a.t.}_n}{B^{a.t.}_j};\ T_{-1}\equiv 0$	1
Auerbach (1991)	$\frac{B^{a.t}_n}{B_n}$	$\frac{B^{a.t.}_n}{B_n}$
Auerbach and Bradford (2004)	$(1-H)\frac{B^{a.t}_n}{B_n}$	$(1-H)\frac{B^{a.t}_n}{B_n}+HB^{a.t.}_n$
Brown (1948)	$1-T$	$(1-T)+TB_n$
Imputed wealth tax	$1-\sum_{p=j}^{n-1}r_pT_p\frac{B^{a.t.}_n}{B^{a.t.}_{p+1}}$	$1-\sum_{p=0}^{n-1}r_pT_p\frac{B^{a.t.}_n}{B^{a.t.}_{p+1}}$
Government takes all risk	0	$1_{\{j=0\}}B^{a.t}_n$

Further comments on these parameterizations and some interpretations of the parameters and relations between the different models are given in the following Examples 2–5. Example 6 is an example in the spirit of Vickrey's original contribution (Vickrey 1939) with an averaging mechanism for the tax base. Example 7 illustrate wash sale adjustments for two of the models, the mark-to-market valuation principle and the retrospective capital gains taxation principle. Finally, Example 8 illustrates the non-neutrality of the usual realization principle.

Example 2 (The mark-to-market valuation principle) The mark-to-market valuation principle is by construction related to the flow of earnings. The role of the bank account term, $B^{a.t.}_n/B^{a.t.}_{j+1}$, in the entry in Table 2 is to transfer the tax payments at different points in time j into their time n equivalents. The identification $k^0_{n,0}=1$ reflects that this taxation principle only depends on the net earnings.

For a fixed T the mark-to-market valuation principle is a convex combination of two other linear taxation systems, cf. item 5 in Theorem 1:

$$K^0_{n,j}=\begin{cases} T\frac{B^{a.t.}_n}{B^{a.t.}_1}, & j=0\\ -r_jT(1-T)\frac{B^{a.t.}_n}{B^{a.t.}_{j+1}}, & j=1,2,\ldots,n-1\\ 1-T, & j=n. \end{cases} \tag{35}$$

The coefficients $K^0_{n,j}$ in each period arise as the imputed wealth tax weighted by $(1-T)$ and the "government takes all risk" tax weighted by T.

The relation between the taxation expressed in terms of earnings and the taxation expressed in terms of stock values is spelled out in the following calculation, which is the special case of (18)–(19) for the mark-to-market valuation principle:

$$\begin{aligned}\text{Wealth after tax at time } n &= \theta^0 S_0 + \sum_{j=1}^{n}\left(1 - T_{j-1}\frac{B_n^{a.t.}}{B_j^{a.t.}}\right)\left(\theta^j S_j - \theta^{j-1} S_{j-1}\right)\\ &= \theta^n S_n - \sum_{j=1}^{n} T_{j-1}\frac{B_n^{a.t.}}{B_j^{a.t.}}\left(\theta^j S_j - \theta^{j-1} S_{j-1}\right)\\ &= (1 - T_{n-1})\theta^n S_n - \sum_{j=1}^{n-1} B_n^{a.t.}\left(\frac{T_j}{B_{j+1}^{a.t.}} - \frac{T_{j-1}}{B_j^{a.t.}}\right)\theta^j S_j\end{aligned} \tag{36}$$

in accordance with the parameters in Tables 1 and 2.

Example 3 (Auerbach's retrospective capital gains tax) Auerbach's retrospective capital gains tax, cf. (Auerbach 1991), is an example of a pure cash flow tax, where taxation only takes place at points in time where there is a cash flow. In Auerbach (1991) taxes are only levied at the horizon date.

The relation between the taxation expressed in terms of earnings and the taxation expressed in terms of stock values is spelled out in the following special case of (18)–(19):

$$\text{Wealth after tax at time } n = \frac{B_n^{a.t.}}{B_n}\theta^n S_n = \frac{B_n^{a.t.}}{B_n}\left[\theta^0 S_0 + \sum_{j=1}^{n}\left(\theta^j S_j - \theta^{j-1} S_{j-1}\right)\right]. \tag{37}$$

Example 4 (The Brown tax and the Auerbach & Bradford tax) The original (Auerbach 1991) retrospective capital gains tax has been generalized in Auerbach and Bradford (2004). This generalized cash flow taxation model is a convex combination of two other schedules, cf. item 5 in Theorem 1. One of them is the Auerbach (1991) system which is the "corner solution" with $H = 0$. The other is the "government takes all risk" taxation system, which is the "corner solution" with $H = 1$.

The Auerbach and Bradford (2004) model is similar in structure to the *pure cash flow tax* or *Brown tax* proposed in Brown (1948). The original Brown tax is an example of the ability to choose different taxation principles for the risk-free part of the income and for the risky part of the income; in Brown (1948) risk-free income is tax free and the bank account grows with the rate of interest before tax.[12] In Auerbach and

[12] The original work of Brown (1948) dealt with investment theory and the design of neutral tax depreciation rules. The tax system was characterized as taxing the return on real investment with a tax base solely based on "non-financial" cash flows.

Bradford (2004) risk-free income is taxed and the bank account after tax grows with the after tax rate of interest.

The relation between the taxation expressed in terms of earnings and the taxation expressed in terms of stock values is spelled out in the following special case of (18)–(19) for the Brown tax:

$$\begin{aligned}\text{Wealth after tax at time } n &= ((1-T)+TB_n)\,\theta^0 S_0 \\ &\quad +(1-T)\sum_{j=1}^{n}\left(\theta^j S_j - \theta^{j-1}S_{j-1}\right) \\ &= (1-T)\theta^n S_n + TB_n\theta^0 S_0. \end{aligned} \tag{38}$$

The similar relation for the Auerbach and Bradford (2004) tax becomes

$$\begin{aligned}\text{Wealth after tax at time } n &= \left((1-H)\frac{B_n^{a.t.}}{B_n} + HB_n^{a.t.}\right)\theta^0 S_0 \\ &\quad +(1-H)\frac{B_n^{a.t.}}{B_n}\sum_{j=1}^{n}\left(\theta^j S_j - \theta^{j-1}S_{j-1}\right) \\ &= (1-H)\frac{B_n^{a.t.}}{B_n}\theta^n S_n + HB_n^{a.t.}\theta^0 S_0. \end{aligned} \tag{39}$$

Hence, the “Brown tax” is equivalent to the Auerbach and Bradford (2004) cash flow taxation system with $H = T$, except for the difference in the taxation of interest earnings.

Example 5 (Imputed wealth tax) The imputed wealth tax is a well known principle applied in, e.g., property taxation schemes. Although it is seemingly unrelated to any earnings based taxation system by construction its representation in terms of the parameters in Table 2 reveals such a component.

$$\begin{aligned}\text{Wealth after tax at time } n &= \left(1-\sum_{p=0}^{n-1} r_p T_p \frac{B_n^{a.t.}}{B_{p+1}^{a.t.}}\right)\theta^0 S_0 \\ &\quad +\sum_{j=1}^{n}\left(1-\sum_{p=j}^{n-1} r_p T_p \frac{B_n^{a.t.}}{B_{p+1}^{a.t.}}\right)\left(\theta^j S_j - \theta^{j-1}S_{j-1}\right) \\ &= \theta^n S_n - \sum_{p=0}^{n-1} r_p T_p \frac{B_n^{a.t.}}{B_{p+1}^{a.t.}}\theta^0 S_0 \\ &\quad -\sum_{p=1}^{n-1}\sum_{j=1}^{p} r_p T_p \frac{B_n^{a.t.}}{B_{p+1}^{a.t.}}\left(\theta^j S_j - \theta^{j-1}S_{j-1}\right)\end{aligned}$$

$$= \theta^n S_n - \sum_{p=0}^{n-1} r_p T_p \theta^p S_p \frac{B_n^{a.t.}}{B_{p+1}^{a.t.}}. \tag{40}$$

Example 6 (A neutral "averaging" tax system) With due respect to the original contribution by Vickrey (1939), consider the following parameters for a generalized linear taxation system, cf. item 3 in Theorem 1:

$$\phi_0 = 1, \quad \phi_j = \frac{1-(j-1)/n}{B_j}, \quad \phi_{n+1} = 0, \quad \psi_j = \frac{1}{nB_j}. \tag{41}$$

This is an example of a generalized version of the Auerbach (1991) taxation system, where the tax base is averaged out over the entire period to depend on all the portfolio values $\theta^j S_j$, $j = 1, 2, \ldots, n$ instead of being dependent solely on the value observed at the horizon. It is easily checked that the system is neutral by plugging into, e.g., the condition in (32).

If the position is realized at time t the investor's after tax revenue is

$$\sum_{j=0}^{t} K_{t,j}^0 \theta^j S_j = \frac{1}{n} \sum_{j=1}^{t-1} \frac{B_t^{a.t.}}{B_j} \theta^j S_j + \left(1 - \frac{t-1}{n}\right) \frac{B_t^{a.t.}}{B_t} \theta^t S_t. \tag{42}$$

The weight $1/n$ is applied to the taxation of each of the terms $\theta^j S_j$, $j = 1, 2, \ldots, t-1$, in accordance with the Auerbach (1991) taxation system which leaves the investor with $(B_j^{a.t.}/B_j)\theta^j S_j$ from the j'th term. This result is carried forward to time t with the bank account after tax, $B_t^{a.t.}/B_j^{a.t.}$. The remaining weight, $(1 - (t-1)/n)$, is applied to the taxation of the term $\theta^t S_t$.

Example 7 (Readjustment due to wash sale) A neutral taxation principle leaves the portfolio *value* invariant to wash sale attempts. If the tax account is settled by taking a short position in the bank account it does not in general leave the portfolio *distribution* invariant, when the same self-financing portfolio strategy is pursued after the wash sale attempt. We work out the details for two of the models mentioned, but the same procedure can be employed for any model.

For the mark-to-market valuation principle we have

$$K_{t,j}^0 = B_t^{a.t.} \left[\frac{T_j}{B_{j+1}^{a.t.}} - \frac{T_{j-1}}{B_j^{a.t.}} \right], \quad K_{t,t}^0 = 1 - T_{t-1} \tag{43}$$

$$A_t^\theta = T_{t-1}\theta^t S_t - B_t^{a.t.} \sum_{j=1}^{t-1} \left[\frac{T_j}{B_{j+1}^{a.t.}} - \frac{T_{j-1}}{B_j^{a.t.}} \right] \theta^j S_j - \frac{B_t^{a.t.}}{B_1^{a.t.}} T_0 \theta^1 S_1. \tag{44}$$

If the tax account is settled by taking a short position in the bank account, and the system is restarted as a result of the wash sale attempt, the final result at time n is

$$(1-T_{n-1})\theta^n S_n + \sum_{j=t+1}^{n-1} \frac{B_n^{a.t.}}{B_t^{a.t.}} \left[\frac{T_j}{\left(B_{j+1}^{a.t.}/B_t^{a.t.}\right)} - \frac{T_{j-1}}{\left(B_j^{a.t.}/B_t^{a.t.}\right)} \right] + \frac{B_n^{a.t.}}{B_{t+1}^{a.t.}} T_t \theta^t S_t$$
$$-\frac{B_n^{a.t.}}{B_t^{a.t.}} T_{t-1}\theta^t S_t + B_n^{a.t.} \sum_{j=1}^{t-1} \left[\frac{T_j}{B_{j+1}^{a.t.}} - \frac{T_{j-1}^{a.t.}}{B_j^{a.t.}} \right] \theta^j S_j + \frac{B_n^{a.t.}}{B_1^{a.t.}} T_0 \theta^1 S_1. \tag{45}$$

After rearranging terms this expression becomes

$$(1-T_{n-1})\theta^n S_n + \sum_{j=1}^{n-1} B_n^{a.t.} \left[\frac{T_j}{B_{j+1}^{a.t.}} - \frac{T_{j-1}}{B_j^{a.t.}} \right] \theta^j S_j + \frac{B_n^{a.t.}}{B_1^{a.t.}} T_0 \theta^1 S_0 \tag{46}$$

which is identical to the result from following the self-financing strategy without the wash sale.

For Auerbach's retrospective capital gains tax model we have

$$K_{t,j}^0 = 0, \quad j \neq t, \qquad K_{t,t}^0 = \frac{B_t^{a.t.}}{B_t} \tag{47}$$

$$K_{n,s}^t = 0, \quad s \neq n, \qquad K_{n,n}^t = \frac{\left(B_n^{a.t.}/B_t^{a.t.}\right)}{(B_n/B_t)}. \tag{48}$$

The Auerbach system taxes the investor by retaining a certain fraction of the wealth. For this neutral taxation system it is easier to finance the settlement of the tax account by realizing part of the portfolio and continue the portfolio policy with the reduced size of the portfolio. When a certain fraction is captured at a realization date $t < n$, the remaining holding period is shortened, so the fraction to be captured at the end of the horizon—out of the remaining wealth—is smaller than it would have been without the wash sale. This makes the sequence of taxation dates irrelevant, as shown in (49):

$$\frac{B_n^{a.t.}}{B_n} = \left(\frac{B_t^{a.t.}}{B_t} \right) \cdot \left(\frac{B_n^{a.t.}/B_t^{a.t.}}{B_n/B_t} \right). \tag{49}$$

However, it also follows if we finance the settlement of the tax account by borrowing. In that case the final result will have the same *value*, but not the same *distribution*, as the final result with no wash sale:

$$\theta^n S_n \frac{B_n^{a.t.}/B_t^{a.t.}}{B_n/B_t} - \frac{B_n^{a.t.}}{B_t^{a.t.}} \theta^t S_t \left(1 - \frac{B_t^{a.t.}}{B_t} \right)$$
$$= B_n^{a.t.} \left[\left(\frac{\theta^n S_n}{B_n} \right) \left(\frac{B_t}{B_t^{a.t.}} \right) + \left(1 - \frac{B_t}{B_t^{a.t.}} \right) \frac{\theta^t S_t}{B_t} \right] \equiv \Psi_n. \tag{50}$$

The value of this position at time 0 is found by using the law of iterated conditional expectations and the fact that $B_t/B_t^{a.t.} \in \mathscr{F}_t$:

$$E^Q\left[\frac{\Psi_n}{B_n^{a.t.}}\right] = E^Q\left[E^Q\left[\frac{\Psi_n}{B_n^{a.t.}}\Big|\mathscr{F}_t\right]\right] = E^Q\left[\frac{\theta^t S_t}{B_t}\right] = \theta^1 S_0. \tag{51}$$

Example 8 (The realization principle and non-neutrality) The realization principle is characterized by the parameter values:

$$K_{nn}^0 = 1 - T, \qquad K_{n0}^0 = T, \qquad K_{nj}^0 = 0 \quad \text{for } j = 1, 2, \ldots, n-1. \tag{52}$$

It is *not* a neutral taxation system. This can be verified by checking, e.g., condition (25):

$$\frac{K_{nn}^0}{B_n^{a.t.}} B_n + \frac{K_{n0}^0}{B_n^{a.t.}} = \frac{(1-T)B_n + T}{B_n^{a.t.}} \stackrel{?}{=} 1. \tag{53}$$

For $n = 1$ this is fulfilled—there is no deferral problem in a one-period context. However, for $n > 1$ we have that[13]

$$\frac{(1-T)B_n + T}{B_n^{a.t.}} > 1. \tag{54}$$

Solving for the "correct" value of K_{n0}^0 in order to make the realization principle neutral shows that the tax value of deducing the initial investment from the tax basis must be less than T:

$$K_{n0}^0 = B_n^{a.t.} - (1-T)B_n < T. \tag{55}$$

It is diminishing with the length of the time horizon and for sufficiently large values of n, K_{n0}^0 even becomes negative! With parameter values $r = 10\%$ and $T = 50\%$ this happens for n close to 15.

Figure 2 shows four curves. The upper curve is the value of the bank account before tax (B_n), and the third curve from above is the value of the bank account after tax ($B_n^{a.t.}$). The curve between these two shows the actual end result from a unit investment in the bank account and having the returns taxed on a realization basis. The lower curve is the value of $K_{n,0}^0$ that corresponds to tax neutrality.

5 Multiple investments, withdrawals and neutral taxation

So far we have assumed that the portfolios were self-financing. This is an important simplification and not only for the theoretical developments. Real life tax codes have

[13] This follows, e.g., from Jensen's inequality applied to the convex function $x \to x^n$ and the arguments $1 + r$ and 1, respectively.

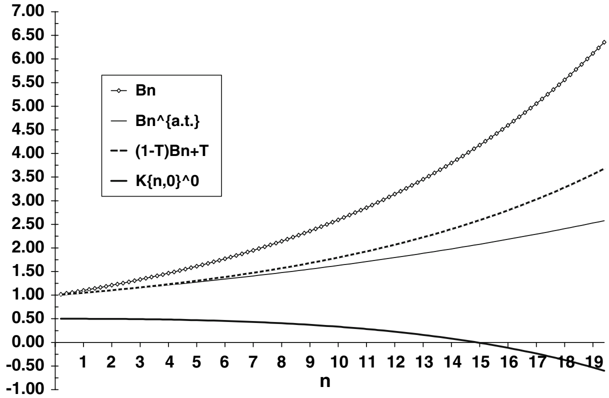

Fig. 2 The value of $K^0_{n,n}$ in a neutral realization based taxation system. Parameter values: $r = 10\%$ and $T = 50\%$

different—and often complicated—rules for how to account for assets that are identical except for the acquisition date; e.g., stocks in the same company bought and sold at different times. Typical examples of such tax rules include

- the assets sold are precisely identified by choice of the investor
- the assets sold are identified by use of the FIFO principle, i.e., assets are identified in chronological order
- the assets sold are identified by use of the LIFO principle, i.e., assets are identified in reverse chronological order and
- the assets sold are *not* identified, but an average acquisition price for the entire position is calculated—and recalculated upon any additional purchase—and used as the basis for calculating taxable capital gains and losses.

Within the class of generalized linear taxation systems it is, of course, a possibility to isolate any new injection of funds as an independent investment on its own. However, this would require a new set of weights to be initiated for every new injection because the holding period must be identified individually; and rules, similar to the ones above, must be determined for withdrawals.

It turns out that there is a simple recipe for handling the tax treatment of portfolios and portfolio changes that are not self-financing. In case of a withdrawal from the portfolio at time t the withdrawn part is simply treated as a realization in accordance with the coefficients $K^0_{t,j}$, $j = 0, 1, \ldots, t$. When a fraction x of the portfolio value is realized at time t, the tax calculations for future realizations are reduced by the factor $1-x$. On the other hand if at time t an additional purchase is made through a capital injection into the portfolio a correction to the tax account A^θ_t is needed. We will determine this correction by use of a variational analysis, taking an existing self-financing portfolio strategy as the point of departure.

Let the capital injection be denoted by $(\Delta\theta)^t S_t$, where the portfolio change due to the capital injection is identified by the symbol Δ. In the time periods after t this capital injection gives rise to changes $(\Delta\theta)^j S_j$, $j = t + 1, \ldots, n$, in the portfolio values. If one ignores the distinction between "existing portfolio" and "capital injection" after time t the condition (21) for neutrality boils down to

$$K^0_{t,t}(\Delta\theta)^t S_t = E^Q\left[\frac{\sum_{j=t}^m K^0_{m,j}(\Delta\theta)^j S_j}{\left(B^{a.t.}_m / B^{a.t.}_t\right)} \,\middle|\, \mathscr{F}_t\right]. \tag{56}$$

Since the investor is investing $(\Delta\theta)^t S_t$, he can be fully compensated for not having the holding period reset by crediting the tax account at time t with the amount $(1 - K^0_{t,t})(\Delta\theta)^t S_t$.

The same insight can be obtained from the dynamics of the tax account as shown in Corollary 1, Eq. (34):

$$\begin{aligned} A^\theta_t &= (1 + r_{t-1}(1 - T_{t-1}))A^\theta_{t-1} + r_{t-1}T_{t-1}\theta^{t-1}S_{t-1} \\ &\quad + \left(1 - K^0_{t,t}\right)\left(\theta^t S_t - (1 + r_{t-1})\theta^{t-1}S_{t-1}\right). \end{aligned} \tag{57}$$

A capital injection $(\Delta\theta)^t S_t$ at time t will change the tax account by $(1 - K^0_{t,t})(\Delta\theta)^t S_t$, which can be neutralized by an offsetting correction of the same size. In Appendix B we show the result of this procedure for the Auerbach (1991) taxation system as well as for the mark-to-market valuation system and verify that the procedure is equivalent in terms of valuation to resetting the holding period. The Brown tax is self-explanatory in this respect.

Now consider the case of a withdrawal, e.g., due to a dividend payment. If dividends are paid out and not reinvested they are taxed precisely as any other realization of a part of the portfolio. The net proceeds after tax from a withdrawal at time t making up the fraction x of the portfolio value is

$$x\sum_{j=0}^{t} K^0_{t,j}\theta^j S_j \tag{58}$$

and the taxes to be paid on future realizations are scaled down by the factor $1-x$ for time indices $j = 0, 1, 2, \ldots, t$. If held until the horizon n the net proceeds after taxes, expressed in terms of the original portfolio strategy θ^j, becomes

$$(1 - x)\sum_{j=0}^{n} K^0_{n,j}\theta^j S_j. \tag{59}$$

Adding to this the proceeds from time t, carried forward to time n, we end up with a total time n proceeds as follows:

$$\sum_{j=0}^{n} K^0_{n,j}\theta^j S_j + x\left[\frac{B^{a.t.}_n}{B^{a.t.}_t}\left(\sum_{j=0}^{t} K^0_{t,j}\theta^j S_j\right) - \sum_{j=0}^{n} K^0_{n,j}\theta^j S_j\right]. \tag{60}$$

It is a straightforward calculation to show that the time t-value of the last part of (60) is zero, whenever Q is an equivalent martingale measure, cf. e.g., (30). This term is

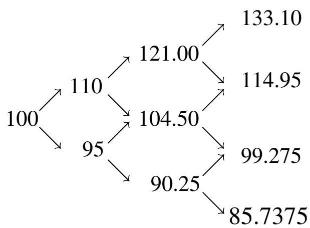

Fig. 3 Binomial lattice

not identically zero due to the implicit risk-free carry forward of the taxes paid at time t, but its time t value is zero. Hence, paying dividends does not affect valuation.

Example 9 Consider a standard binomial model as illustrated in Fig. 3.

Assume that it is decided to pay out dividends in periods 1 and 2 equal to 4% of the value of the investment in this asset and realize the position at time 3. Denoting the stochastic values in the lattice in Fig. 3 as (S_1, S_2, S_3), the payment stream to an investor, who is investing 100 at time 0, becomes

$$(-100, 0.04 \cdot S_1, 0.04 \cdot 0.96 \cdot S_2, 0.96^2 \cdot S_3). \tag{61}$$

In this way there are two consecutive values of 4% for x.

There are eight different paths through this lattice, and we denote the movements through the lattice by "u" and "d" for up- and down-movements, respectively.

The cash flows before tax as well as after tax from this investment are shown in Table 3. We consider how the scheme in (58)–(59) works for two of the neutral taxation systems considered, the mark-to-market valuation system and the Auerbach (1991) retrospective capital gains taxation system. We use the tax rate $T = 40\%$, and the rate of interest r is set to 5%. Hence, $r^{a.t.} = 3\%$.

As shown in the last line of Table 3, both systems produce cash flows with the same present value for each of the three periods and the same present values as for the before tax cash flow. Table 3 also illustrates the different degrees of risk sharing between the investor and the government for these two systems.

Assume alternatively that the dividends are reinvested and that the net tax bill to be paid at time t is borrowed in the bank account. According to the scheme outlined the reinvestment would lead to an immediate tax benefit of the magnitude $x(1 - K^0_{t,t})\theta^t S_t$; hence, the net tax payment at time t becomes

$$-\sum_{j=0}^{t-1} K^0_{t,j} x \theta^j S_j. \tag{62}$$

Repeating the carry forward argument the taxes to be paid at time n under this reinvestment scenario become

Table 3 Cash flows with 4% dividend payments

Path	Cash flow before tax			Cash flow after tax					
				Mark-to-market			Auerbach (1991)		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
uuu	4.4	4.6464	122.6650	4.24	4.3192	110.1168	4.3162	4.4711	115.7882
uud	4.4	4.6464	105.9379	4.24	4.3192	100.0806	4.3162	4.4711	99.9989
udu	4.4	4.0128	105.9379	4.24	3.9391	100.2631	4.3162	3.8614	99.9989
udd	4.4	4.0128	91.4918	4.24	3.9391	91.5954	4.3162	3.8614	86.3627
duu	3.8	4.0128	105.9379	3.88	3.9460	100.4339	3.7276	3.8614	99.9989
dud	3.8	4.0128	91.4918	3.88	3.9460	91.7663	3.7276	3.8614	86.3627
ddu	3.8	3.4656	91.4918	3.88	3.6177	91.9239	3.7276	3.3348	86.3627
ddd	3.8	3.4656	79.0157	3.88	3.6177	84.4382	3.7276	3.3348	74.5860
Present value	4.00	3.8400	92.1600	4.00	3.8400	92.1600	4.0000	3.8400	92.1600

$$\left((1 - K^0_{n,n})\theta^n S_n - \sum_{j=t}^{n-1} K^0_{n,j}\theta^j S_j \right) - (1-x) \sum_{j=0}^{t-1} K^0_{n,j}\theta^j S_j. \tag{63}$$

Subtracting the carry forward value of the taxes paid at time t, cf. (62), leads to the net result at time n:

$$\begin{aligned}
&\left(K^0_{n,n}\theta^n S_n + \sum_{j=t}^{n-1} K^0_{n,j}\theta^j S_j \right) + (1-x) \sum_{j=0}^{t-1} K^0_{n,j}\theta^j S_j + \frac{B^{a.t.}_n}{B^{a.t.}_t} \sum_{j=0}^{t-1} K^0_{t,j} x \theta^j S_j \\
&= \sum_{j=0}^{n} K^0_{n,j}\theta^j S_j - x \left(\sum_{j=0}^{t-1} \left(K^0_{n,j} - \frac{B^{a.t.}_n}{B^{a.t.}_t} K^0_{t,j} \right) \theta^j S_j \right).
\end{aligned} \tag{64}$$

Because of the relations (22) each individual term in the latter sum is zero. Hence, these derivations prove the holding period neutrality of the tax mechanism proposed for partial realizations—including wash sales—and dividend payments.

6 Summary and conclusion

The concept of a neutral taxation system is a normative benchmark with which other taxation systems may be compared and the severeness of the deviations measured. A neutral taxation system has the property that all investors, irrespective of their individual tax situation, agree on the market prices and that attempts to exploit wash sale opportunities in order to generate tax deferrals are ruled out.

This paper investigates the consequences of the "no arbitrage" assumption applied to after tax values. We provide necessary and sufficient conditions that characterize in an exhaustive manner when a linear and symmetric taxation system is a neutral taxation system. This is done in a fairly general multi-period framework, and the analysis points out in what sense and to what extent interest rates and tax rates can be stochastic. The results in the paper do not depend in any way on any specific equilibrium model properties. They are directly in line with the no arbitrage paradigm which builds on a minimum requirement for an asset pricing model, namely that arbitrage opportunities do not exist.

For the *general version* of Theorem 1 we need to have a given finite time horizon. This is a weakness when one thinks in terms of implementation along the lines of the tax rules discussed here. Fortunately, this requirement is not needed for any of the specific examples given in Table 1, where the mechanisms can be extended to any horizon.

Appendix A: Proofs

Proof The set of attainable claims is the subspace spanned by the two vectors (S_u, S_d) and $(1, 1)$:[14]

$$\mathscr{M} \equiv \left\{ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \,\middle|\, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \theta_1 \begin{pmatrix} S_u \\ S_d \end{pmatrix} + \theta_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ (\theta_0, \theta_1) \in \mathbb{R}^2 \right\}. \tag{A.1}$$

When $K \neq 0$ this is obviously the same subspace as the one spanned by the vectors

$$K \begin{pmatrix} S_u - (1+r)S_0 \\ S_d - (1+r)S_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S_0 \\ S_0 \end{pmatrix}.$$

However, when $K = 0$ the only after tax values attainable are the values proportional to the vector $(1, 1)$. Hence, the set of attainable claims is invariant to a linear and symmetric taxation schedule for $K \neq 0$. □

Proof (Theorem 1) The important parts of Theorem 1 are statements 1. and 2. and the conditions in (29). These provide necessary and sufficient conditions on the parameters of the taxation system in order to ensure valuation neutrality.

The statement in 3. is merely a rewriting of—and, hence, equivalent to—the second of the conditions in (29). In the same way the statements in 4. are restatements of the conditions in (29) once it is observed that the tax account is reset to value zero at any intermediate realization. After such an intermediate realization the clock as well as the discounting are both restarted.

The statement in 5. is a trivial consequence of the conditions in 1. Since a given linear and symmetric neutral taxation schedule is uniquely characterized by the sequence $\left(K^0_{n,0}, K^0_{n,1}, \ldots, K^0_{n,n}\right)$ it follows from (29) that any convex combination of two such

[14] This subspace is trivially $\mathbb{R}^2$ in this simple setup, but for completeness a formal proof is given.

schedules is again a linear and symmetric neutral taxation schedule.[15] It can also be deduced directly from the basic requirement in (30).

The statement in 6. is directly related to the statement in 2., which can be interpreted as an "invertibility property", and the proof is similar. To the extent that the risk exposure after tax is not nullified completely, it is possible to switch between the after-tax and the before-tax relations.

As for 1. consider the representation (21) which is equivalent to (30). Then the first half of the conditions in (29) is obvious—the first sum in (21) vanishes except for the term $j = t$. It remains to show that the rest vanishes due to the second part of the conditions in (29). This follows from the following sequence of calculations:

$$\frac{\theta^t S_t}{B^t} = E^Q\left[\frac{\theta^n S_n}{B^n}\,\middle|\,\mathscr{F}_t\right] \wedge \frac{K^0_{t,t}}{B^{a.t.}_t} = \sum_{j=t}^{n} \frac{K^0_{n,j}}{B^{a.t.}_n}\frac{B_j}{B_t} \quad \Rightarrow \tag{A.2}$$

$$\frac{K^0_{t,t}}{B^{a.t.}_t}\theta^t S_t = E^Q\left[\sum_{j=t}^{n} \frac{K^0_{n,j}}{B^{a.t.}_n} B_j \frac{\theta^n S_n}{B^n}\,\middle|\,\mathscr{F}_t\right] \quad \Rightarrow \tag{A.3}$$

$$\frac{K^0_{t,t}}{B^{a.t.}_t}\theta^t S_t = \sum_{j=t}^{n} E^Q\left[E^Q\left[\frac{K^0_{n,j}}{B^{a.t.}_n} B_j \frac{\theta^n S_n}{B^n}\,\middle|\,\mathscr{F}_j\right]\middle|\,\mathscr{F}_t\right] \tag{A.4}$$

$$= \sum_{j=t}^{n} E^Q\left[\frac{K^0_{n,j}}{B^{a.t.}_n} B_j E^Q\left[\frac{\theta^n S_n}{B_n}\,\middle|\,\mathscr{F}_j\right]\middle|\,\mathscr{F}_t\right] \tag{A.5}$$

$$= \sum_{j=t}^{n} E^Q\left[\frac{K^0_{n,j}}{B^{a.t.}_n} B_j \frac{\theta^j S_j}{B_j}\,\middle|\,\mathscr{F}_t\right] \tag{A.6}$$

$$= \sum_{j=t}^{n} E^Q\left[\frac{K^0_{n,j}}{B^{a.t.}_n}\theta^j S_j\,\middle|\,\mathscr{F}_t\right]. \tag{A.7}$$

This ends the proof of 1. □

In order to prove 2. an induction argument is applied. First, consider an investment strategy such that $\theta^1 S_0 \neq 0$. For $m = 1$, (30) takes on the form

$$\theta^1 S_0 = \frac{K^0_{10}}{B^{a.t.}_1}\theta^1 S_0 + \frac{K^0_{11}}{B^{a.t.}_1} E^Q[\theta^1 S_1]. \tag{A.8}$$

For the special investment strategy $\theta^1 S_1 = \theta^1 S_0 B_1$, i.e., investing in the bank account, (30) becomes

$$\theta^1 S_0 = \frac{K^0_{10}}{B^{a.t.}_1}\theta^1 S_0 + \frac{K^0_{11}}{B^{a.t.}_1}\theta^1 S_0 B_1 \quad \Leftrightarrow \quad 1 = \frac{K^0_{10}}{B^{a.t.}_1} + \frac{K^0_{11}}{B^{a.t.}_1} B_1. \tag{A.9}$$

[15] As a matter of fact in pure mathematical terms an affine combination is sufficient. However, this might result in taxation schedules that are economically meaningless.

We have made use if the fact that $B_1 \in \mathscr{F}_0$. Given this relation, we can substitute into (A.8) and get

$$0 = \frac{K_{11}^0}{B_1^{a.t.}} B_1 \left(\theta^1 S_0 - E^Q \left[\frac{\theta^1 S_1}{B_1} \right] \right). \tag{A.10}$$

Whenever $K_{11}^0 \neq 0$ the pricing relation before tax must be fulfilled with the equivalent martingale measure Q. This establishes the beginning of an induction proof with $t = 0$ and $m = 1$.

Next, consider the case $m = 2$. For any self-financing portfolio strategy we must have the pricing relation for $t = 1$:

$$\begin{aligned} \frac{K_{10}^0}{B_1^{a.t.}} \theta^1 S_0 + \frac{K_{11}^0}{B_1^{a.t.}} \theta^1 S_1 &= E^Q \left[\frac{K_{20}^0}{B_2^{a.t.}} \theta^1 S_0 + \frac{K_{21}^0}{B_2^{a.t.}} \theta^1 S_1 + \frac{K_{22}^0}{B_2^{a.t.}} \theta^2 S_2 \middle| \mathscr{F}_1 \right] \\ &= \frac{K_{20}^0}{B_2^{a.t.}} \theta^1 S_0 + \frac{K_{21}^0}{B_2^{a.t.}} \theta^1 S_1 + \frac{K_{22}^0}{B_2^{a.t.}} E^Q \left[\theta^2 S_2 \mid \mathscr{F}_1 \right] \quad \Leftrightarrow \end{aligned} \tag{A.11}$$

$$\left(\frac{K_{10}^0}{B_1^{a.t.}} - \frac{K_{20}^0}{B_2^{a.t.}} \right) \theta^1 S_0 = \left(\frac{K_{21}^0}{B_2^{a.t.}} - \frac{K_{11}^0}{B_1^{a.t.}} \right) \theta^1 S_1 + \frac{K_{22}^0}{B_2^{a.t.}} E^Q \left[\theta^2 S_2 \mid \mathscr{F}_1 \right]. \tag{A.12}$$

The following is a special case of relation (A.12), corresponding to an investment strategy with $\theta^2 S_2 = \theta^1 S_1 (B_2 / B_1)$:

$$\left(\frac{K_{10}^0}{B_1^{a.t.}} - \frac{K_{20}^0}{B_2^{a.t.}} \right) \theta^1 S_0 = \left(\frac{K_{21}^0}{B_2^{a.t.}} + \frac{K_{22}^0}{B_2^{a.t.}} \frac{B_2}{B_1} - \frac{K_{11}^0}{B_1^{a.t.}} \right) \theta^1 S_1. \tag{A.13}$$

We also have the pricing relation for $t = 0$:

$$\theta^1 S_0 = E^Q \left[\frac{K_{20}^0}{B_2^{a.t.}} \theta^1 S_0 + \frac{K_{21}^0}{B_2^{a.t.}} \theta^1 S_1 + \frac{K_{22}^0}{B_2^{a.t.}} \theta^2 S_2 \right]. \tag{A.14}$$

Assume that a self-financing investment strategy can be constructed such that $\theta^1 S_0 = 0$, but $\theta^1 S_1 \neq 0 \ \forall \omega \in \Omega$. This is verified below. Then the following identity between random variables immediately follows from (A.13):

$$\frac{K_{11}^0}{B_1^{a.t.}} = \frac{K_{21}^0}{B_2^{a.t.}} + \frac{K_{22}^0}{B_2^{a.t.}} \frac{B_2}{B_1}. \tag{A.15}$$

Choosing any alternative investment strategy with $\theta^1 S_0 \neq 0$ leads to

$$\frac{K_{20}^0}{B_2^{a.t.}} = \frac{K_{10}^0}{B_1^{a.t.}}. \tag{A.16}$$

This proves the necessity of the relations (29) for $m = 2$ and $t = 0, 1$. Additionally, we can rewrite (A.11) after cancellation as

$$0 = \frac{K^0_{22}}{B^{a.t.}_2} B_2 \left[E^Q \left[\frac{\theta^2 S_2}{B_2} \middle| \mathscr{F}_1 \right] - \frac{\theta^1 S_1}{B_1} \right]. \tag{A.17}$$

When $K^0_{22} \neq 0$ we have the equivalent before tax relation

$$\frac{\theta^1 S_1}{B_1} = E^Q \left[\frac{\theta^2 S_2}{B_2} \middle| \mathscr{F}_1 \right]. \tag{A.18}$$

The regularity condition—that a non-trivial investment strategy exists—means that at any atom $A \in \mathscr{F}_t$ it is possible to find a portfolio such that the realized return at any succeeding atom $B \in \mathscr{F}_{t+1},\ B \subseteq A$, is never equal to the risk-free rate of return. This also implies that it is always possible to find a portfolio with zero net investment such that the realized value after one period is never zero in any state. The recipe is to combine a non-trivial investment strategy at time t with a short position of identical value in the bank account. When $t = 0$ the result of such an investment strategy is never zero in any state at time 1. Or, stated alternatively, an investment strategy always exists such that $\theta^1 S_0 = 0$, but $\theta^1 S_1 \neq 0\ \forall \omega \in \Omega$. Similar reasoning can be applied at any point in time t by simply investing in nothing until time t.

Assume now that the necessity is established for (m, t), $m = 0, 1, \ldots, \bar{m}$, $t = 0, 1, \ldots, m$. Consider the relation (30) for neighboring indices $\bar{m}$ and $\bar{m} + 1$:

$$\begin{aligned} \frac{1}{B^{a.t.}_{\bar{m}}} \sum_{j=0}^{\bar{m}} K^0_{\bar{m},j} \theta^j S_j &= E^Q \left[\frac{1}{B^{a.t.}_{\bar{m}+1}} \sum_{j=0}^{\bar{m}+1} K^0_{\bar{m}+1,j} \theta^j S_j \middle| \mathscr{F}_{\bar{m}} \right] \\ &= \frac{1}{B^{a.t.}_{\bar{m}+1}} \left(\sum_{j=0}^{\bar{m}} K^0_{\bar{m}+1,j} \theta^j S_j + K^0_{\bar{m}+1,\bar{m}+1} E^Q \left[\theta^{\bar{m}+1} S_{\bar{m}+1} \middle| \mathscr{F}_{\bar{m}} \right] \right). \end{aligned} \tag{A.19}$$

Following the same procedure as above we can find a self-financing portfolio strategy such that

$$\theta^1 S_0 = \theta^1 S_1 = \cdots = \theta^{\bar{m}-1} S_{\bar{m}-1} = 0, \quad \theta^{\bar{m}} S_{\bar{m}} \neq 0, \quad \theta^{\bar{m}+1} S_{\bar{m}+1} = \theta^{\bar{m}} S_{\bar{m}} \frac{B_{\bar{m}+1}}{B_{\bar{m}}}. \tag{A.20}$$

This establishes the necessity of the relation

$$\frac{K^0_{\bar{m},\bar{m}}}{B^{a.t.}_{\bar{m}}} = \frac{K^0_{\bar{m}+1,\bar{m}}}{B^{a.t.}_{\bar{m}+1}} + \frac{K^0_{\bar{m}+1,\bar{m}+1}}{B^{a.t.}_{\bar{m}+1}} \frac{B_{\bar{m}+1}}{B_{\bar{m}}}. \tag{A.21}$$

By backwards substitution one can generate all the desired relations in the second half of (29). Additionally, when $K^0_{\bar{m}+1,\bar{m}+1} \neq 0$ the pre-tax relation, cf. (A.18), follows along the lines of reasoning in (A.17):

$$\frac{\theta^{\bar{m}} S_{\bar{m}}}{B_{\bar{m}}} = E^Q \left[\frac{\theta^{\bar{m}+1} S_{\bar{m}+1}}{B_{\bar{m}+1}} \,\middle|\, \mathscr{F}_{\bar{m}} \right]. \tag{A.22}$$

Working backwards step by step and in turn construct investment strategies in accordance with the recipe in (A.20) leads to the necessity of the relations

$$\frac{K^0_{\bar{m},j}}{B^{a.t.}_{\bar{m}}} = \frac{K^0_{\bar{m}+1,j}}{B^{a.t.}_{\bar{m}+1}} \tag{A.23}$$

for $j = \bar{m} - 1,\ \bar{m} - 2,\ \ldots,\ 1, 0$. This ends the induction proof of 2.

Finally, we prove statement 6. We want to construct a self-financing trading strategy Ψ^t in such a way that the resulting wealth after tax is identical to the wealth before tax for a given self-financing trading strategy θ^t. We use the notation:

$$W_t \equiv \theta^t S_t, \quad \widehat{W}^{a.t.}_t = \Psi^t S_t - A^{\Psi}_t. \tag{A.24}$$

Solving for the dynamics in (33), cf. also (12), we obtain:

$$W_t = B_t W_0 + \sum_{j=1}^{t} \frac{B_t}{B_j} \left(W_j - (1 + r_{j-1}) W_{j-1} \right) \quad \Leftrightarrow \tag{A.25}$$

$$\widehat{W}^{a.t.}_t = B^{a.t.}_t W_0 + \sum_{j=1}^{t} \frac{K^0_{j,j} B^{a.t.}_t}{B^{a.t.}_j} \left[W_j - (1 + r_{j-1}) W_{j-1} \right]. \tag{A.26}$$

We want to find a trading strategy Ψ^t such that $\widehat{W}_t = W_t$. Then the value before tax evolves as $B_s E^Q \left[\frac{W_t}{B_t} \middle| \mathscr{F}_s \right]$, whereas the value after tax evolves as $B^{a.t.}_s E^Q \left[\frac{W_t}{B^{a.t.}_t} \middle| \mathscr{F}_s \right]$.

If B_t as well as $B^{a.t.}_t$ evolve in a deterministic way the solution is straightforward. By investing $\left(B_t / B^{a.t.}_t \right) W_0$ in a bank account and entering into the sequence of zero net investment transactions with portfolio weights

$$\Psi^j = \frac{1}{K_{j,j}} \frac{B_t / B_j}{B^{a.t.}_t / B^{a.t.}_j} \theta^j \tag{A.27}$$

it is clear by inspection that the final result is W_t. When comparing the portfolio policy Ψ^j with θ^j observe that the replicating strategy for a taxed investor is subjected to two magnification effects: one from the risk sharing mechanism $(1/K_{jj})$ and one from the difference in accumulation factor on the bank accounts before and after tax over the remaining time horizon $\left(\frac{B_t / B_j}{B^{a.t.}_t / B^{a.t.}_j} \right)$.

If B_t and/or $B_t^{a.t.}$ evolve stochastically an induction argument must be invoked. We rewrite (A.25) and (A.26) as

$$W_t = \frac{B_t}{B_{t-1}} W_{t-1} + \theta^t (S_t - (1 + r_{t-1}) S_{t-1}) \tag{A.28}$$

$$W_t^{a.t.} = \frac{B_t^{a.t.}}{B_{t-1}^{a.t.}} W_{t-1}^{a.t.} + K_{tt} \Psi^t (S_t - (1 + r_{t-1}) S_{t-1}) \tag{A.29}$$

we see that by choosing $\Psi_t \equiv \theta^t / K_{tt}$ the last two terms become identical. As for the first term assume we use as the induction assumption that the set of attainable claims after tax at time $t-1$ is identical to the set of attainable claims before tax. This means that the claim $(1 + r_{t-1})/(1 + r_{t-1}(1 - T_{t-1}))W_{t-1}$ is attainable after tax. Hence, the claim that the attainable sets before tax and after tax are identical is true for t when it is true for $t-1$.

We only need to show that the sets are identical for $t = 1$ in order to complete the induction argument. But this is trivial, since for $t = 1$ the relations (A.25)–(A.26) become

$$W_1 = (1 + r_0) W_0 + \theta^1 (S_1 - (1 + r_0) S_0) \tag{A.30}$$

$$\widehat{W}_1 = (1 + r_0(1 - T_0)) \widehat{W}_0 + K_{11} \Psi^1 (S_1 - (1 + r_0) S_0). \tag{A.31}$$

Hence, the magnification effect already described works for this case, which completes the proof.

Appendix B: Capital injections

In this appendix some details from Sect. 5 are written out.

We verify—for two key models—that the proposed tax crediting mechanism for incremental capital injections has the same effect in terms of valuation as resetting the clock for the holding period.

For the Auerbach (1991) model a unit investment at time t gives rise to a tax credit of $1 - B_t^{a.t.}/B_t$. At the horizon date n this unit investment has value $(\Delta\theta)^n S_n$, which is taxed in accordance with the tax parameters originating at time 0. The value at time t of this position—including the tax benefit—is one, equal to the capital injection at time t:

$$E^Q \left[\frac{(\Delta\theta)^n S_n \frac{B_n^{a.t.}}{B_n}}{\left(B_n^{a.t.}/B_t^{a.t.}\right)} + \left(1 - \frac{B_t^{a.t.}}{B_t}\right) \middle| \mathscr{F}_t \right] = \frac{B_t^{a.t.}}{B_t} E^Q \left[\frac{(\Delta\theta)^n S_n}{B_n / B_t} \middle| \mathscr{F}_t \right] + \left(1 - \frac{B_t^{a.t.}}{B_t}\right) = 1. \tag{B.32}$$

For the mark-to-market valuation principle the tax benefit should be T_{t-1}. I.e., we want to prove that

$$E^Q\left[\frac{(1-T_{n-1})(\Delta\theta)^n S_n + \sum_{j=t}^{n-1}\left[\frac{T_j}{B_{j+1}^{a.t.}} - \frac{T_{j-1}}{B_j^{a.t.}}\right] B_n^{a.t.}(\Delta\theta)^j S_j}{B_n^{a.t.}/B_t^{a.t.}} + T_{t-1}\middle|\mathscr{F}_t\right] = 1. \tag{B.33}$$

We prove this by backwards induction. For $j = n-1$ we have the following result:

$$\begin{aligned}
&E^Q\left[\frac{(1-T_{n-1})(\Delta\theta)^n S_n + T_{n-1}(\Delta\theta)^{n-1}S_{n-1} - \frac{T_{n-2}B_n^{a.t.}}{B_{n-1}^{a.t.}}(\Delta\theta)^{n-1}S_{n-1}}{B_n^{a.t.}/B_t^{a.t.}}\middle|\mathscr{F}_t\right] \\
&= E^Q\left[E^Q\left\{\frac{(1-T_{n-1})(\Delta\theta)^n S_n + T_{n-1}(\Delta\theta)^{n-1}S_{n-1} - \frac{T_{n-2}B_n^{a.t.}}{B_{n-1}^{a.t.}}(\Delta\theta)^{n-1}S_{n-1}}{B_n^{a.t.}/B_t^{a.t.}}\middle|\mathscr{F}_{n-1}\right\}\middle|\mathscr{F}_t\right] \\
&= E^Q\left[\frac{(\Delta\theta)^{n-1}S_{n-1} - T_{n-2}(\Delta\theta)^{n-1}S_{n-1}}{B_{n-1}^{a.t.}/B_t^{a.t.}}\middle|\mathscr{F}_t\right].
\end{aligned} \tag{B.34}$$

This is so because by construction taxation by the mark-to-market valuation principle for a self-financing portfolio leads to the relation

$$E^Q\left[\frac{(1-T_{t-1})(\Delta\theta)^t S_t + T_{t-1}(\Delta\theta)^{t-1}S_{t-1}}{B_t^{a.t.}}\middle|\mathscr{F}_{t-1}\right] = \frac{(\Delta\theta)^{t-1}S_{t-1}}{B_{t-1}^{a.t.}}. \tag{B.35}$$

Next include $j = n-2$ to obtain

$$E^Q\left[\frac{(1-T_{n-3})(\Delta\theta)^{n-2}S_{n-2}}{B_{n-2}^{a.t.}/B_t^{a.t.}}\middle|\mathscr{F}_t\right]. \tag{B.36}$$

This continues until $j = t$ where by assumption $(\Delta\theta)^t S_t = 1$. Hence, the expression is reduced to

$$E^Q\left[\frac{(1-T_{t-1})(\Delta\theta)^t S_t}{B_t^{a.t.}/B_t^{a.t.}} + T_{t-1}\middle|\mathscr{F}_t\right] = (1-T_{t-1})\cdot 1 + T_{t-1} = 1. \tag{B.37}$$

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